de Rham's Perfect Sneak

M^{c}Kiernan

In the 1950's de Rham wrote several papers investigating functions defined by functional equations. The following presents one such function; when we first learned of this function in the late fifties, we invented the name "perfect sneak"—however this name is not necessarily known to anyone other than myself and my fellow students way back then.

Consider a biased coin with probability p of heads, with $0 < p < 1$. The player starts with x dollars—for simplicity let us assume $0 \leq x \leq 1$, although this is not necessary. The player repeatedly flips the coin, winning with probability p and losing with probability $1 - p$ each time. The ultimate goal of this game is for the player to eventually acquire $x = 1$ dollar, subject to the following rule:

> when the player has $x \leq 1/2$ dollars, he must bet all he has, *i.e.* x: hence if he loses on this toss, he clearly has no money left and hence loses the game, while if he wins on this toss he now has $2x$ dollars and plays again;
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> when the player has $x \geq 1/2$ dollars, he must bet $1 - x$: hence if he wins the toss he wins the game (has one dollar) but if he loses, he still has $x - (1 - x) = 2x - 1$ dollars and can play again.

Let $f(x)$ be the probability that, starting with x dollars, our player will eventually attain one dollar. Trivially

$$f(0) = 0, \quad f(1) = 1 \tag{1}$$

while for any given p, $0 < p < 1$ (note: $p \neq 0, p \neq 1$),

$$0 \leq x \leq 1/2 \quad \Rightarrow \quad f(x) = p \cdot f(2x) \; + \; (1-p) \cdot f(0) \tag{2}$$

$$1/2 \leq x \leq 1 \quad \Rightarrow \quad f(x) = p \cdot f(1) \quad + \; (1-p) \cdot f(2x-1) \tag{3}$$

In view of the interpretation of $f(x)$, one would conjecture

$$f \text{ is strictly monotonic increasing on } [0,1], \tag{4}$$

$$f \text{ is continuous on } [0,1], \tag{5}$$

$$f(x) = x \text{ when } p = 1/2. \tag{6}$$

In fact (6) can be verified immediately—(1), (2), (3) hold trivially. Another possible conjecture would be that even when $p \neq 1/2$, the solution of such a simple game should be reasonably simple. Quite the contrary; while (4), (5), and (6) will be verified, we also have

$$f'(x) = 0 \text{ whenever the derivative exists, if } p \neq 1/2 \tag{7}$$

(and since strictly monotonic, $f'(x)$ exists a.e., except for a set of measure zero, on [0,1]). The function "sneaks" up monotonically, in fact strictly monotonically, from $f(0) = 0$ to $f(1) = 1$ and $f' = 0$ a.e.; this explains the curious name we gave de Rham's function.*

The solution of (1), (2), and (3) is given as follows:

for each $0 \leq x \leq 1$ write x in the binary expansion, *i.e.*

$$x = \sum_{i=0}^{\infty} \frac{1}{2^{\alpha_i}} \quad \text{for integer } \alpha_0 < \alpha_1 < \alpha_2 < \ldots \tag{8}$$

*The Cantor step function also sneaks from 0 to 1 continuously, but this function is constant on intervals, not *strictly* monotonic.

and define

$$f(x) = \sum_{i=0}^{\infty} \left(\frac{1-p}{p}\right)^i \cdot p^{\alpha_i}, \quad 0 < p < 1, \quad x \neq 0. \tag{9}$$

That (9) converges for all $0 < p < 1$ and integer $0 \leq \alpha_0 < \alpha_1 < ...$ is a simple ratio test problem. We must verify (1), (2) and (3). Since

$$x = 1 = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \quad \Rightarrow \quad f(1) = \sum_{i=0}^{\infty} (1-p)^i \cdot p = \frac{p}{1-(1-p)} = 1 \tag{10}$$

we have (1) in view of (10). Further

for $0 < x < 1/2$, $\alpha_0 \geq 1$ and $2x = \sum_{i=0}^{\infty} \frac{1}{2^{\alpha_i - 1}}$, implying

$$\begin{aligned} f(2x) = \sum_{i=0}^{\infty} \left(\frac{1-p}{p}\right)^i \cdot p^{\alpha_i - 1} &= \frac{1}{p} \cdot \sum_{i=0}^{\infty} \left(\frac{1-p}{p}\right)^i \cdot p^{\alpha_i} \\ &= \frac{1}{p} f(x), \end{aligned}$$

which is (2) in view of (10);

for $1/2 \leq x < 1$, $\alpha_0 = 1$ and $x = 1/2 + \sum_{i=1}^{\infty} \frac{1}{2^{\alpha_i}}$, implying

$$2x - 1 = \sum_{i=1}^{\infty} \frac{1}{2^{\alpha_i - 1}} = \sum_{i=0}^{\infty} \frac{1}{2^{\alpha_{i+1} - 1}},$$

and hence

$$\begin{aligned} f(2x-1) = \sum_{i=0}^{\infty} \left(\frac{1-p}{p}\right)^i p^{\alpha_{i+1} - 1} &= \frac{1}{1-p} \cdot \sum_{i=0}^{\infty} \left(\frac{1-p}{p}\right)^{i+1} \cdot p^{\alpha_{l+1}} \\ = \frac{1}{1-p} \cdot \sum_{i=1}^{\infty} \left(\frac{1-p}{p}\right)^i \cdot p^{\alpha_i} &= \frac{1}{1-p} \cdot \left\{ f(x) - \left(\frac{1-p}{p}\right)^0 p^{\alpha_0} \right\} \end{aligned}$$

and since $\alpha_0 = 1$,

$$(1-p) f(2x-1) = f(x) - p$$

which is (3), since $f(1) = 1$.

The remaining details are less trivial—the reader interested in the proof is referred to Lajos Takács, *An increasing continuous singular function*, Amer. Math. Monthly, January 1978, pp 35–37. (Note p here is Takács' $\frac{1}{1+\rho}$, $0 < \rho < \infty$.)

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$$f(2x) = \sum_{i=0}^{\infty} \left(\frac{1-p}{p}\right)^i \cdot p^{\alpha_i - 1} = \frac{1}{p} \cdot \sum_{i=0}^{\infty} \left(\frac{1-p}{p}\right)^i \cdot p^{\alpha_i}$$

$$= \frac{1}{p} f(x), \text{ which is (2) in view of (10);}$$

for $\frac{1}{2} \le x < 1$, $\alpha_0 = 1$ and $x = \frac{1}{2} + \sum_{i=1}^{\infty} \frac{1}{2^{\alpha_i}}$, implying

$$2x - 1 = \sum_{i=1}^{\infty} \frac{1}{2^{\alpha_i - 1}} = \sum_{i=0}^{\infty} \frac{1}{2^{\alpha_{i+1} - 1}}, \text{ and hence}$$

$$f(2x-1) = \sum_{i=0}^{\infty} \left(\frac{1-p}{p}\right)^i p^{\alpha_{i+1} - 1} = \frac{1}{1-p} \cdot \sum_{i=0}^{\infty} \left(\frac{1-p}{p}\right)^{i+1} \cdot p^{\alpha_{i+1}}$$

$$= \frac{1}{1-p} \cdot \sum_{i=1}^{\infty} \left(\frac{1-p}{p}\right)^i \cdot p^{\alpha_i} = \frac{1}{1-p} \cdot \left\{ f(x) - \left(\frac{1-p}{p}\right)^0 p^{\alpha_0} \right\}$$

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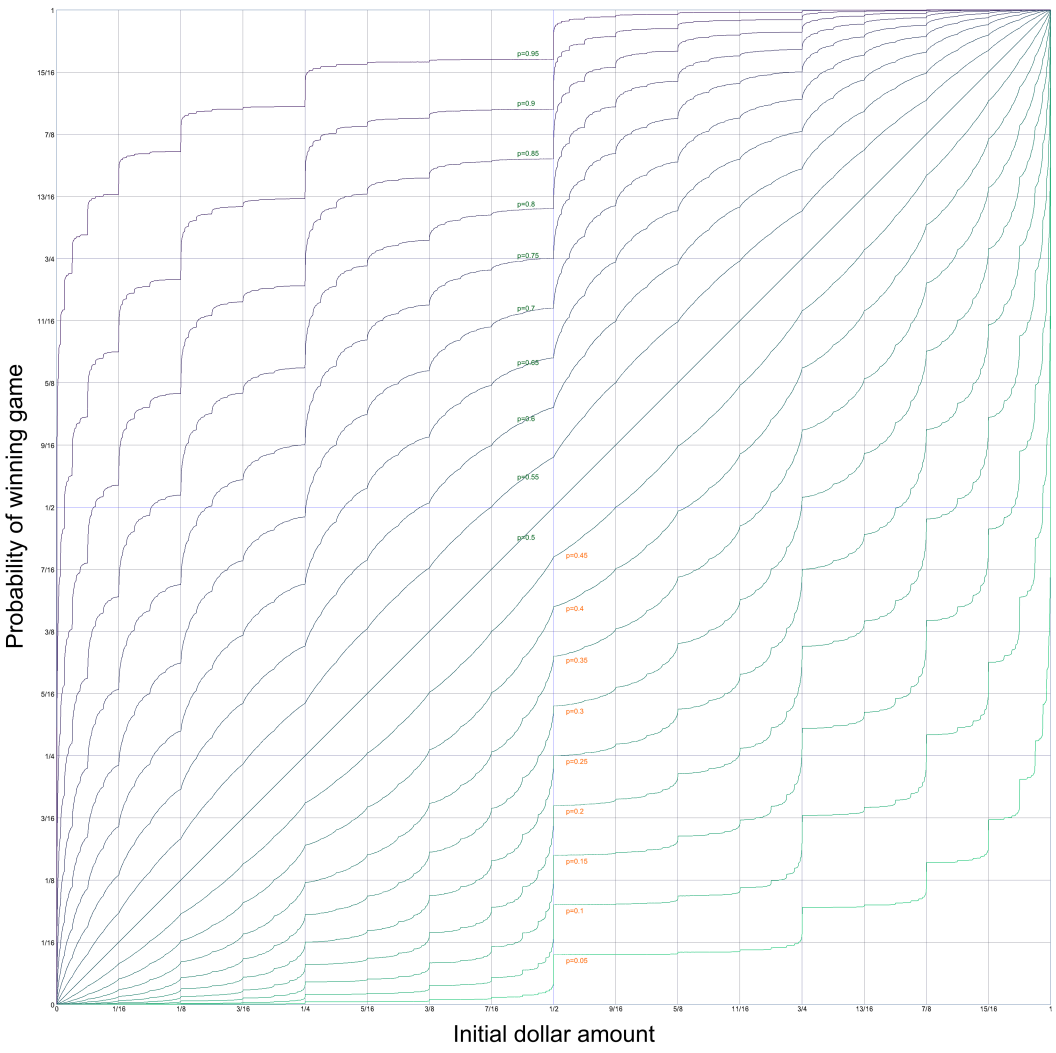

Figure 1: Examples of $f(x)$ for the coin bias (odds of winning a toss) p from 0.05 to 0.95, labeled as such. Each curve was estimated by playing the game 10^9 times for that p. Note the kinks at reciprocal powers of two.

```
import random

coin_bias = 0.9   # a.k.a. p, 0 < p < 1
num_points = 1000

dollars_start = 0.0
dollars_step = 1.0 / (num_points-1)
while dollars_start < 1.0:

    num_wins = 0
    num_games = 100000
    for k in range(0, num_games):

        dollars = dollars_start
        while 0.0 < dollars and dollars < 1.0:
            if dollars < 0.5:
                bet = dollars
            else:
                bet = 1.0 - dollars

            coin_toss = random.random()   # in [0,1)
            if coin_toss < coin_bias:
                dollars += bet
            else:
                dollars -= bet

        if dollars >= 1.0:
            num_wins += 1

    print(dollars_start, num_wins / num_games)   # avg. number of wins
    dollars_start += dollars_step
```

Listing 1. A Python program that calculates one of the curves in Fig. 1. To increase accuracy at the cost of slower calculation, increase `100000`.

This transcription of Mike McKiernan's 1986 handout for MATH 140B, advanced calculus, was made by three of his students, Kevyn Collins-Thompson, David Schilling, and Camille Goudeseune, all B.Math. 1989 U. of Waterloo, in 2026. At that time they were unaware of his passing, and they mourn his loss.

In 2000, Berg and Krüppel analyzed this function's self-similarity and other properties in *De Rham's singular function and related functions*, Z. Anal. Anwend. 19(1):227–237.

Georges de Rham (1903–1990) studied under Lebesgue, Cartan, and other luminaries. He taught at Lausanne and Geneva, and researched differential forms, cohomology theory, and differential topology.

Michel Amedee "Mike" McKiernan (1930–2007) studied with Karl Menger at the Illinois Institute of Technology, earning his Ph.D. in 1956. His best known paper is *The functional differential equation* $Df = 1/ff$, Proc. AMS 8(2):230–233 (1957). Others are *On the* n*th derivative of composite functions*, AMM 63(5):331–333 (1956), and *On iteration groups and related functional equations*, Aeq. Math. 31:37–46 (1986). In 1967 he was one of the founders of the U. of Waterloo's department of Applied Analysis and Computer Science, which became its School of Computer Science. He served on the committee of the 1984 Canadian Mathematical Olympiad. One of his Waterloo advisees, Bev Marshman, herself became an academic advisor there.